

Virasoro Algebra and $O(N)$ σ -model

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Abstract

Within the framework of a local expansion of the logarithm of the $O(N)$ σ -model vacuum functional, valid for slowly varying fields, the modified Virasoro algebra is studied. The operator-like central charge term is given, up to second order, for a general functional.

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I. INTRODUCTION

During the last twenty years a great deal of work has been done on conformally symmetric theories. Importantly, they can be exactly solved to give critical exponents of two dimensional theories, aiding to their classification. Moreover, their conformal symmetry enables the association of strongly interacting fields with weakly coupled ones, which are easy to elaborate. To make physically interesting theories out of them it is necessary to incorporate interactions (e.g. curvature) in the free case. The cases of interacting fields which preserve the conformal symmetry in a stronger or a weaker sense have been studied in the literature [1], as well as theories where their interactions destroy this symmetry. One of the latter is the non-linear $O(N)$ σ -model, where the $O(N)$ symmetry generates a mass term in the quantum level, destroying the classical conformal symmetry of the model.

A considerable amount of interest is concentrated on the cylindrical space-time $\mathbf{R}^1 \times S^1$, which shares many features with string theory [2]. We will face one of them, the Virasoro algebra, which has been previously studied through different quantization procedures, by using the functional formalism. Our aim is to set up a general formalism for the study of a modified form of the Virasoro algebra for the $O(N)$ σ -model, for which instead of the usual central charge term we expect operator-like terms as a quantum anomaly extension of this algebra. As the form of the vacuum functional is not known, we will calculate the anomaly extension up to an undefined function on which the operator-like term acts. We expect that further study of the renormalization group properties of the model [3], will determine completely this term up to the approximation scheme adopted in the following. Note that in this scheme we expand the vacuum functional for configurations with small momentum (i.e. large distances), where phenomena of mass generation are dominant.

II. VIRASORO ALGEBRA

Let us briefly review the Virasoro Algebra for string theory. We can have a string on an N dimensional manifold parameterized by X^μ while the string spans a $1 + 1$ dimensional Minkowski world sheet with coordinates σ and τ . The position of the string on the manifold is given by the functions $X^\mu(\sigma, \tau)$. The string can be open or closed. In the second case we demand $X(\sigma, \tau)$ to be periodic in σ , which we assume runs around the string in the interval $[-\pi, \pi]$, while in the first case σ takes values in the interval $[0, \pi]$. The action of the string can be written

$$S = -\frac{1}{2\pi} \int d\sigma d\tau \eta^{\alpha\beta} \partial_\alpha X \cdot \partial_\beta X , \quad (1)$$

which is the action for a σ -model, now chosen to be in a flat background manifold. In the Hamiltonian formalism $P = \dot{X}$ ($\dot{\cdot}$ stands for a derivative with respect to time) and at the quantum level the action (1) is accompanied by the commutation relations

$$[X^\mu(\sigma), P_\nu(\sigma')] = i\delta_\nu^\mu \delta(\sigma - \sigma') , \quad [X^\mu(\sigma), X_\nu(\sigma')] = [P^\mu(\sigma), P_\nu(\sigma')] = 0 \quad (2)$$

For the case of the closed string there are two oscillating modes going “left” and “right”. These modes can be generated by the Virasoro operators

$$L_n = \frac{1}{4} \int_{-\pi}^{\pi} d\sigma : (P^\mu - X'^\mu)(P_\mu - X'_\mu) : e^{-in\sigma}$$

$$\tilde{L}_m = \frac{1}{4} \int_{-\pi}^{\pi} d\sigma : (P^\mu + X'^\mu)(P_\mu + X'_\mu) : e^{im\sigma}$$

for m and n integers different in general, that is the two modes are independent from each other. The operators

$$\alpha_n^\mu \equiv \frac{1}{\sqrt{\pi}} \int d\sigma e^{-in\sigma} (P^\mu(\sigma) - X'^\mu(\sigma)) \quad \text{and} \quad \tilde{\alpha}_n^\mu \equiv \frac{1}{\sqrt{\pi}} \int d\sigma e^{in\sigma} (P^\mu(\sigma) + X'^\mu(\sigma)) \quad (3)$$

play the role of the creation operators for $n < 0$ and the annihilation ones for $n > 0$ with respect to the vacuum state, $|0\rangle$, of the string. α and $\tilde{\alpha}$ satisfy the commutation relations

$$[\alpha_n^\mu, \alpha_m^\nu] = \eta^{\mu\nu} n \delta_{m+n,0} = [\tilde{\alpha}_n^\mu, \tilde{\alpha}_m^\nu] , \quad [\alpha_n^\mu, \tilde{\alpha}_m^\nu] = 0 \quad (4)$$

following from (2). We can re-write the Virasoro operators as

$$L_n = \frac{1}{2} \sum : \alpha_{n+p}^\mu \alpha_{-p}^\nu : \eta_{\mu\nu} \quad \text{and} \quad \tilde{L}_n = \frac{1}{2} \sum : \tilde{\alpha}_{n+p}^\mu \tilde{\alpha}_{-p}^\nu : \eta_{\mu\nu} , \quad (5)$$

where now the meaning of the normal ordering symbol $::$ is clearly defined with respect to the operators α and $\tilde{\alpha}$ acting on the vacuum state, $|0\rangle$. It is actually needed only for the case $n = 0$, as $L_0 + \tilde{L}_0 + 2$ represents the Hamiltonian of the closed string and needs normalization of the vacuum energy (performed here by normal ordering). We can check that the normal ordered Virasoro operators satisfy the well-known Virasoro algebra

$$[L_n, L_m] = (n - m)L_{n+m} + \frac{N}{12} \delta_{n,-m} (n^3 - n) . \quad (6)$$

The same algebra is also satisfied by the \tilde{L} operators, and by the Virasoro operators of the open string. In [4], an algebraic derivation of (6) is presented.

III. THE VACUUM STATE OF THE STRING

We can construct the wave functional $\langle X|0\rangle$, that represents the vacuum state $|0\rangle$, explicitly by using the annihilation relation $\alpha_{n\mu}|0\rangle = 0$ for $n > 0$. The equation it has to satisfy is

$$\alpha_{n\mu} \langle X|0\rangle = 0 \Rightarrow \frac{1}{\sqrt{\pi}} \int d\sigma e^{-in\sigma} \left(-i \frac{\delta}{\delta X^\mu(\sigma)} - X'_\mu(\sigma) \right) \langle X|0\rangle = 0 , \quad (7)$$

where the momentum operator has been represented by $P_\mu(\sigma) = -i\frac{\delta}{\delta X^\mu(\sigma)}$. From (7) we see that the vacuum state will have a Gaussian localized form. Let us take

$$\langle X|0\rangle = \exp\left(\iint d\sigma d\sigma' X^\mu(\sigma)H(\sigma,\sigma')X_\mu(\sigma')\right)$$

where $H(\sigma,\sigma')$ is a symmetric function of σ and σ' , to be calculated. As all the points on the closed string are equivalent, H should be a function of the difference $\sigma - \sigma'$. So its decomposition in modes will be

$$H(\sigma,\sigma') = H(\sigma - \sigma') = \sum_{m=-\infty}^{\infty} H^m e^{im(\sigma-\sigma')}$$

where H^m are the Fourier components of $H(\sigma,\sigma')$. After substituting into (7) we obtain $H^m = -\frac{m}{4\pi}$ for $m \geq 0$. As $H(\sigma,\sigma')$ has to be symmetric in its arguments we finally have

$$H^m = -\frac{|m|}{4\pi}, \quad \text{for any integer } m.$$

IV. FUNCTIONAL METHOD FOR THE CENTRAL CHARGE TERM

We can define the regularized Virasoro operators as

$$L[u] = \frac{1}{4} \iint d\sigma d\sigma' u(\sigma,\sigma') G_s^{\mu\nu}(\sigma,\sigma') [P_\mu(\sigma) - X'_\mu(\sigma)] [P_\nu(\sigma') - X'_\nu(\sigma')] \equiv \quad (8)$$

$$\frac{1}{4} \iint d\sigma d\sigma' u(\sigma,\sigma') G_s^{\mu\nu}(\sigma,\sigma') R_\mu(\sigma) R_\nu(\sigma'),$$

where $G_s^{\mu\nu}(\sigma,\sigma')$ is a Kernel to point split the double action of the functional differentiations, satisfying the condition $\lim_{s \rightarrow 0} G_s^{\mu\nu}(\sigma,\sigma') = \eta^{\mu\nu} \delta(\sigma,\sigma')$, $R_\mu(\sigma) \equiv P_\mu(\sigma) - X'_\mu(\sigma)$, $u(\sigma,\sigma')$ is the component of a vector field on the circle, S^1 , on which $X(\sigma)$ is defined and is symmetric in σ and σ' . For $u(\sigma,\sigma') = 1$ we get the divergent quantity $L[1]$, which is equivalent to the divergent Hamiltonian of the string.

We will calculate the commutator $[L[u], L[v]]$ acting on the vacuum state $\langle X|0\rangle$ given in the previous section. We have

$$[L[u], L[v]] \langle X|0\rangle =$$

$$\frac{1}{16} \iint d\sigma d\sigma' d\bar{\sigma} d\bar{\sigma}' u(\sigma,\sigma') v(\bar{\sigma},\bar{\sigma}') G_s^{\mu\nu}(\sigma,\sigma') G_s^{\kappa\lambda}(\bar{\sigma},\bar{\sigma}') \times$$

$$[R_\mu(\sigma) R_\nu(\sigma'), R_\kappa(\bar{\sigma}) R_\lambda(\bar{\sigma}')] \langle X|0\rangle =$$

$$-\frac{1}{4} \iint d\sigma d\sigma' d\bar{\sigma} d\bar{\sigma}' u(\sigma,\sigma') v(\bar{\sigma},\bar{\sigma}') G_s^{\mu\nu}(\sigma,\sigma') G_s^{\kappa\lambda}(\bar{\sigma},\bar{\sigma}') \times$$

$$i\eta_{\mu\kappa} \delta'(\sigma,\bar{\sigma}) \{R_\nu(\sigma') R_\lambda(\bar{\sigma}') + R_\lambda(\bar{\sigma}') R_\nu(\sigma')\} \langle X|0\rangle. \quad (9)$$

(9) results after applying the relation $[R_\mu(\sigma), R_\kappa(\bar{\sigma})] = -2i\delta'(\sigma,\bar{\sigma})\eta_{\mu\kappa}$ and re-arranging the σ variables as well as the indices. By commuting $R_\lambda(\bar{\sigma}')$ and $R_\nu(\sigma')$ and integrating by

parts, we get

$$\begin{aligned}
& [L[u], L[v]] \langle X|0 \rangle = \\
& \frac{i}{2} \iint d\sigma d\sigma' \left\{ \frac{\partial}{\partial \sigma} u(\sigma, \sigma') v(\sigma, \sigma) G_s^{\lambda\nu}(\sigma, \sigma') + \right. \\
& \left. u(\sigma, \sigma') v(\sigma, \sigma) \frac{\partial}{\partial \sigma} G_s^{\lambda\nu}(\sigma, \sigma') \right\} R_\nu(\sigma') R_\lambda(\sigma) \langle X|0 \rangle
\end{aligned} \tag{10}$$

where the rest of the terms give zero, as they appear symmetric in u and v or include the quantity $\frac{\partial}{\partial \sigma} G_s^{\mu\nu}(\sigma, \sigma') \Big|_{\sigma'=\sigma}$, which is zero. Applying the combination $R_\nu(\sigma') R_\lambda(\sigma)$ on the vacuum state $\langle X|0 \rangle$ the only ambiguity would be from the action of the two functional derivatives. That is

$$\begin{aligned}
& \frac{\delta}{\delta X^\mu(\sigma')} \frac{\delta}{\delta X^\nu(\sigma)} \langle X|0 \rangle = \\
& \left[4 \int H(\sigma', \sigma'') X_\mu(\sigma'') d\sigma'' \int H(\sigma, \sigma'') X_\nu(\sigma'') d\sigma'' + 2H(\sigma, \sigma') \eta_{\mu\nu} \right] \langle X|0 \rangle .
\end{aligned} \tag{11}$$

As the Kernel acts on (11) the only divergency will come from the last term in the square brackets. The other terms will be combined as in the normal ordered case to give the normalized Virasoro operator appearing on the r.h.s. of the algebra. Let us study the term

$$\begin{aligned}
T \equiv & -i \iint d\sigma d\sigma' \left\{ \frac{\partial}{\partial \sigma} u(\sigma, \sigma') v(\sigma, \sigma) G_s^{\mu\nu}(\sigma, \sigma') + \right. \\
& \left. u(\sigma, \sigma') v(\sigma, \sigma) \frac{\partial}{\partial \sigma} G_s^{\lambda\nu}(\sigma, \sigma') \right\} \eta_{\lambda\nu} H(\sigma, \sigma') .
\end{aligned} \tag{12}$$

We can take the Kernel to be of the form

$$G_s^{\mu\nu}(\sigma, \sigma') = \eta^{\mu\nu} \mathcal{G}_s(\sigma, \sigma') = \eta^{\mu\nu} \frac{e^{-(\sigma-\sigma')^2/4s}}{\sqrt{2\pi s}} , \tag{13}$$

where

$$\lim_{s \rightarrow 0} \frac{e^{-(\sigma-\sigma')^2/4s}}{\sqrt{2\pi s}} = \delta(\sigma - \sigma') .$$

Expression (12) can be symmetrized with respect to σ and σ' so that the summation in H can be re-written as

$$\begin{aligned}
H(\sigma, \sigma') &= -\frac{1}{4\pi} \sum_{m=-\infty}^{\infty} |m| e^{im(\sigma-\sigma')} \rightarrow \\
&\rightarrow -\frac{2}{4\pi} \sum_{m=1}^{\infty} m e^{im(\sigma-\sigma')} = -\frac{1}{2\pi i} \sum_{m=1}^{\infty} \frac{\partial}{\partial \sigma} e^{im(\sigma-\sigma')} = -\frac{1}{2\pi i} \frac{\partial}{\partial \sigma} \left(\frac{1}{1 - e^{i(\sigma-\sigma')}} \right)
\end{aligned}$$

Substituting this into (12) we get

$$\begin{aligned}
T &= -\frac{iN}{2} \iint d\sigma d\sigma' \left\{ \frac{\partial}{\partial \sigma} u(\sigma, \sigma') v(\sigma, \sigma) \mathcal{G}_s(\sigma, \sigma') + \right. \\
& \left. u(\sigma, \sigma') v(\sigma, \sigma) \frac{\partial}{\partial \sigma} \mathcal{G}_s(\sigma, \sigma') \right\} \frac{-1}{2\pi i} \frac{\partial}{\partial \sigma} \left\{ \frac{1}{1 - e^{i(\sigma-\sigma')}} - \frac{1}{1 - e^{-i(\sigma-\sigma')}} \right\} .
\end{aligned} \tag{14}$$

By integrating by parts the σ derivative acting on the kernel, we get

$$T = \frac{N}{4\pi} \iint d\sigma d\sigma' 2 \frac{\partial}{\partial \sigma} u(\sigma, \sigma') v(\sigma, \sigma) \mathcal{G}_s(\sigma, \sigma') \frac{\partial}{\partial \sigma} \left\{ \frac{1}{1 - e^{i(\sigma - \sigma')}} - \frac{1}{1 - e^{-i(\sigma - \sigma')}} \right\} - \frac{N}{4\pi} \iint d\sigma d\sigma' u(\sigma, \sigma') v(\sigma, \sigma) \mathcal{G}_s(\sigma, \sigma') \frac{\partial^2}{\partial \sigma^2} \left\{ \frac{1}{1 - e^{i(\sigma - \sigma')}} - \frac{1}{1 - e^{-i(\sigma - \sigma')}} \right\} . \quad (15)$$

We can define $x = \sigma - \sigma'$. In (15) as s gets small the kernel, $\mathcal{G}_s(x)$, becomes nonzero only for small x , so that we can expand the ratio as

$$\frac{1}{1 - e^{ix}} = \frac{i}{x} + \frac{1}{2} - \frac{i}{12}x + \dots$$

Assuming the σ dependence of u to be of the form $u(\sigma, \sigma') \equiv u(\frac{\sigma + \sigma'}{2})$, we can make the following expansions

$$u(\sigma, \sigma') = u(\sigma - \frac{x}{2}) = u(\sigma) - \frac{\partial u(\sigma)}{\partial \sigma} \frac{x}{2} + \frac{\partial^2 u(\sigma)}{\partial \sigma^2} \frac{x^2}{2^2 2!} + \frac{\partial^3 u(\sigma)}{\partial \sigma^3} \frac{x^3}{2^3 3!} + \dots \quad (16)$$

and

$$\frac{\partial}{\partial \sigma} u(\sigma, \sigma') = \frac{1}{2} \frac{\partial u(\sigma)}{\partial \sigma} - \frac{\partial^2 u(\sigma)}{\partial \sigma^2} \frac{x}{2^2} + \frac{\partial^3 u(\sigma)}{\partial \sigma^3} \frac{x^2}{2^4} + \dots \quad (17)$$

At the limit $x \rightarrow 0$ the significant nonzero terms in (15) are

$$T = \frac{N}{4\pi} \iint d\sigma dx v(\sigma, \sigma) \mathcal{G}_s(x) \left\{ \frac{\partial u(\sigma)}{\partial \sigma} \left(-\frac{2i}{x^2} \right) + \frac{\partial u(\sigma)}{\partial \sigma} \left(-\frac{i}{6} \right) + \frac{1}{2^3} \frac{\partial^3 u(\sigma)}{\partial \sigma^3} (-2i) + \left(-\frac{1}{2} \right) \frac{\partial u(\sigma)}{\partial \sigma} \left(-\frac{4i}{x^2} \right) - \frac{\partial^3 u(\sigma)}{\partial \sigma^3} \frac{-4i}{2^3 3!} + \dots \right\} = -\frac{Ni}{24\pi} \int d\sigma \left\{ \frac{\partial u(\sigma)}{\partial \sigma} + \frac{\partial^3 u(\sigma)}{\partial \sigma^3} \right\} v(\sigma) . \quad (18)$$

We can see that the rest of the terms in (10), apart from T , construct the renormalized part of

$$i \iint d\sigma d\sigma' \frac{\partial}{\partial \sigma} u(\sigma, \sigma') v(\sigma, \sigma) G_s^{\nu\lambda}(\sigma, \sigma') R_\nu(\sigma') R_\lambda(\sigma) \langle X|0 \rangle \Big|_{renorm}$$

and as we can antisymmetrize in u and v we have

$$\frac{i}{2} \iint d\sigma d\sigma' \left\{ \frac{\partial}{\partial \sigma} u(\sigma, \sigma') v(\sigma, \sigma) - u(\sigma, \sigma) \frac{\partial}{\partial \sigma} v(\sigma, \sigma') \right\} \times$$

$$G_s^{\nu\lambda}(\sigma, \sigma') R_\nu(\sigma') R_\lambda(\sigma) \langle X|0 \rangle \Big|_{renorm} =$$

$$-iL[[u, v]] \langle X|0 \rangle \Big|_{renorm}$$

for $[u, v] = uv' - u'v$, by using (17). This is the desired result as it is calculated in [5] with another method. We see that the commutator $[L[u], L[v]]$ gives the renormalized part of $-iL[[u, v]]$ plus a finite constant without any infinities appearing in. For $u = e^{-in\sigma}$ and $v = e^{-im\sigma}$ it gives (6).

V. APPLICATIONS TO THE $O(N)$ σ -MODEL

A similar treatment can be applied to the $O(N)$ σ -model, where we expect the corresponding algebra to differ from the Virasoro algebra, because the β_g function is nonzero. In other words this model quantized is not conformally invariant. Operators defined in a similar way as the Virasoro ones for a conformal theory will satisfy an algebra with a different central charge term in structure (e.g. it could be a function rather than a constant) and in origin, as these operators are no longer generators of a conformal transformation. Now σ parameterizes the infinite space-line, so there will be “left” and “right” modes, as in the case of the closed string. We will use the form (8) for our generalized Virasoro operators, where here the coordinates $X(\sigma) \equiv z(\sigma)$ represent a manifold with $O(N)$ symmetry (see [7] for conventions and definitions of the $O(N)$ σ -model). We can work with operators which have similar form to the Hamiltonian, Lorentz and momentum operator. By studying their algebra we can connect them with the generalized Virasoro operators and deduce the algebra the latter satisfy. Instead of a general vector $u(\sigma)$, we can use powers of the σ variable (i.e. σ^k , $k \geq 0$). For a covariant functional derivative given from

$$\frac{D}{Dz^\mu(\sigma)} V^\nu(\sigma') = \frac{\delta V^\nu(\sigma')}{\delta z^\mu(\sigma)} + \int d\sigma'' \Gamma_{\mu\rho}^\nu(\sigma, \sigma', \sigma'') V^\rho(\sigma'') , \quad (19)$$

where $\Gamma_{\mu\rho}^\nu(\sigma, \sigma', \sigma'') = \delta(\sigma - \sigma') \delta(\sigma' - \sigma'') \Gamma_{\mu\rho}^\nu(z(\sigma))$, when e.g. acting on a local vector $V^\mu(\sigma')$, we can make the following definitions

$$\begin{aligned} P &\equiv -i \int \int d\sigma d\sigma' G_s^{\mu\nu}(\sigma, \sigma') g_{\nu\kappa} z'^\kappa \frac{D}{Dz^\mu(\sigma)} \equiv -i \int d\sigma \mathcal{P}(\sigma) \\ H &\equiv \frac{1}{2} \left[- \int \int d\sigma d\sigma' G_s^{\mu\nu}(\sigma, \sigma') \frac{D}{Dz^\mu(\sigma)} \frac{D}{Dz^\nu(\sigma')} + \int d\sigma g_{\mu\nu}(\sigma) z'^\mu z'^\nu \right] \equiv \frac{1}{2} \int d\sigma \mathcal{H} \\ L^k &\equiv \frac{1}{2} \left[- \int \int d\sigma d\sigma' \left(\frac{\sigma + \sigma'}{2} \right)^k G_s^{\mu\nu}(\sigma, \sigma') \frac{D}{Dz^\mu(\sigma)} \frac{D}{Dz^\nu(\sigma')} + \right. \\ &\quad \left. \int \int d\sigma d\sigma' \left(\frac{\sigma + \sigma'}{2} \right)^k g_{\mu\nu}(\sigma) z'^\mu z'^\nu \right] \equiv M^k + \tilde{\mathcal{B}} \equiv \frac{1}{2} \int d\sigma \sigma^k \mathcal{H} \\ \mathcal{B} &\equiv \int d\sigma g_{\mu\nu}(\sigma) z'^\mu z'^\nu \quad \text{and} \quad \tilde{\mathcal{B}} \equiv \int d\sigma \sigma^k g_{\mu\nu}(\sigma) z'^\mu z'^\nu , \end{aligned}$$

where $\tilde{\mathcal{B}}$ could be described with two σ integrations connected by the Kernel G instead of the metric g and the term σ^k written as $(\sigma + \sigma')^k / 2^k$. However, this will only differ from the given expression by terms of order $O(s^n)$ with $n > 0$, which will vanish when the limit $s \rightarrow 0$ is taken. The kernel $G_s^{\mu\nu}(\sigma, \sigma') \equiv \mathcal{G}_s(\sigma, \sigma') K^{\mu\nu}(\sigma, \sigma'; s)$, where $K^{\mu\nu}(\sigma, \sigma'; s)$, with $K^{\mu\nu}(\sigma, \sigma; 0) = g^{\mu\nu}(\sigma)$, is a function of z and its form can be derived after considering rotational symmetry and Poincaré invariance (see [7], [8] for its form up to second order). We can take, as in [8], $\mathcal{G}_s(\sigma, \sigma) = \mathcal{G}_s(0) = \frac{b_0^0}{\sqrt{s}}$, $\mathcal{G}_s''(0) = \frac{b_0^1}{\sqrt{s^3}}$ and $\mathcal{G}_s'''(0) = \frac{b_0^2}{\sqrt{s^5}}$. Now the

commutator of H and L^k is

$$\begin{aligned}
& \left[\int d\sigma \mathcal{H}, \int d\sigma \sigma^k \mathcal{H} \right] \Psi = \\
& \left[- \int d\sigma \bar{\Delta}(\sigma) + \mathcal{B}, - \int d\sigma \sigma^k \bar{\Delta}(\sigma) + \tilde{\mathcal{B}} \right] \Psi = \\
& \left[- \int d\sigma \bar{\Delta}, - \int d\sigma \sigma^k \bar{\Delta} \right] \Psi + \left[\mathcal{B}, - \int d\sigma \sigma^k \bar{\Delta} \right] \Psi + \left[- \int d\sigma \bar{\Delta}, \tilde{\mathcal{B}} \right] \Psi . \tag{20}
\end{aligned}$$

The second term becomes

$$\begin{aligned}
& \left[\mathcal{B}, - \int d\sigma \sigma^k \bar{\Delta} \right] \Psi = \\
& \int d\sigma \sigma^k \left((\bar{\Delta} \mathcal{B}) \Psi + 2G_s^{\mu\nu} \frac{D}{Dz^\mu} \mathcal{B} \frac{D}{Dz^\nu} \Psi \right) = \\
& k(k-1) \frac{b_0^0}{\sqrt{s}} N \int d\sigma \sigma^{k-2} - 2N \frac{b_0^1}{\sqrt{s}^3} \int d\sigma \sigma^k - \\
& \iint d\sigma d\sigma' \left(\frac{\sigma + \sigma'}{2} \right)^k 2G_s^{\mu\nu}(\sigma, \sigma') (-g_{\gamma\mu} 2\mathcal{D}z'^\gamma)|_\sigma \frac{D\Psi}{Dz^\nu(\sigma')} ,
\end{aligned}$$

while the third

$$\begin{aligned}
& \left[- \int d\sigma \bar{\Delta}, \tilde{\mathcal{B}} \right] \Psi = \\
& - \int d\sigma \left[(\bar{\Delta} \tilde{\mathcal{B}}) \Psi + 2G_s^{\mu\nu} \frac{D\tilde{\mathcal{B}}}{Dz^\mu} \frac{D\Psi}{Dz^\nu} \right] = \\
& 0 - \iint d\sigma d\sigma' 2G_s^{\mu\nu}(\sigma, \sigma') \left(-2g_{\gamma\mu} k \sigma^{k-1} z'^\gamma - 2g_{\gamma\mu} \sigma^k \mathcal{D}z'^\gamma \right) \frac{D\Psi}{Dz^\nu(\sigma')} .
\end{aligned}$$

The second and third terms together give

$$\begin{aligned}
& 4 \frac{k(k-1)}{4} \frac{b_0^0}{\sqrt{s}} N \int d\sigma \sigma^{k-2} \Psi - 4 \frac{2N}{4} \frac{b_0^1}{\sqrt{s}^3} \int d\sigma \sigma^k \Psi + 4 \int d\sigma k \sigma^{k-1} \mathcal{P}(\sigma) \Psi = \\
& 4 \int d\sigma k \sigma^{k-1} \mathcal{P}(\sigma) \Psi \tag{21}
\end{aligned}$$

as the first and second terms vanish under the antisymmetry of the commutator with respect to $(\frac{\sigma+\sigma'}{2})^k$ and 1. The previous result is independent of the choice of the functional Ψ . However, the first term of (20) depends on the specific form of Ψ . We can take Ψ , as in the previous section, to be the vacuum wave functional. It is possible to construct the vacuum functional as a derivative expansion of its logarithm in local functionals (see [6], [7]). Working in the lowest order of approximation for slowly varying fields (small momentum) the vacuum functional becomes $\Psi = e^{\int f_{\mu\nu} z'^\mu z'^\nu}$, where $f_{\mu\nu}$ is an ultra-local function which can be determined from the Schrödinger equation. The following calculations will be performed

for up to two derivatives with respect to σ . For example, the action of the two functional derivatives on Ψ gives

$$\frac{\delta^2}{\delta z(\sigma)\delta z(\sigma')} e^{\int f_{\mu\nu} z'^{\mu} z'^{\nu}} = \frac{\delta^2 \int f_{\mu\nu} z'^{\mu} z'^{\nu}}{\delta z(\sigma)\delta z(\sigma')} e^{\int f_{\mu\nu} z'^{\mu} z'^{\nu}} + \frac{\delta \int f_{\mu\nu} z'^{\mu} z'^{\nu}}{\delta z(\sigma)} \frac{\delta \int f_{\mu\nu} z'^{\mu} z'^{\nu}}{\delta z(\sigma')} e^{\int f_{\mu\nu} z'^{\mu} z'^{\nu}}$$

from which we only consider the first term, as the second involves four derivatives with respect to σ . So we only need to consider the $\int f_{\mu\nu} z'^{\mu} z'^{\nu}$ functional in the expansion of the exponential Ψ . For this case we have

$$M^k \int f_{\mu\nu} z'^{\mu} z'^{\nu} =$$

$$\int d\sigma \sigma^k \left\{ (J_2 f)_{\mu\nu} z'^{\mu} z'^{\nu} + \frac{1}{s} b_0^0 f_{\nu}^{\nu} \right\} + \int d\sigma k \sigma^{k-1} 2b_0^0 D^{\nu} f_{\nu\mu} z'^{\mu} + \int d\sigma \frac{k(k-1)}{4} \sigma^{k-2} b_0^0 2f_{\nu}^{\nu} ,$$

also

$$\Delta M^k \int f_{\mu\nu} z'^{\mu} z'^{\nu} =$$

$$.. + \int d\sigma k \sigma^{k-1} 2b_0^0 (J_1 D^{\nu} f_{\nu})_{\mu} z'^{\mu} + \int d\sigma \frac{k(k-1)}{4} \sigma^{k-2} b_0^0 2(J_0 f_{\nu}^{\nu})$$

and

$$M^k \Delta \int f_{\mu\nu} z'^{\mu} z'^{\nu} =$$

$$.. + \int d\sigma k \sigma^{k-1} 2b_0^0 D^{\nu} (J_2 f)_{\nu\mu} z'^{\mu} + \int d\sigma k(k-1) \sigma^{k-2} b_2 (J_2 f)_{\nu}^{\nu} ,$$

where

$$\begin{aligned} (J_n f)_{\rho_1 \dots \rho_n} &= b_0^0 g^{\mu\nu} \left(D_{\mu} D_{\nu} f_{\rho_1 \dots \rho_n} + n f_{\lambda(\rho_2 \dots \rho_n} R_{\rho_1) \mu\nu}^{\lambda} \right) \\ &\quad - n(n-1) \left(b_1^1 g^{\mu\nu} f_{\mu\nu(\rho_3 \dots \rho_n} g_{\rho_1 \rho_2)} + b_2^1 f_{\rho_1 \dots \rho_n} \right) , \end{aligned} \quad (22)$$

and “..” represent homogeneous terms, which do not contribute to the commutator. Using the identities

$$(\text{tr } J_n f)_{\rho_3 \dots \rho_n} - (J_{n-2} \text{tr } f)_{\rho_3 \dots \rho_n} = 0$$

$$(D \cdot J_n f) - J_{n-1} (D \cdot f)_{\rho_2 \dots \rho_n} = 0$$

from [7], when we asked for the Poincaré algebra to be satisfied ($k=1$), we get

$$(\Delta M^k - M^k \Delta) \int f_{\mu\nu} z'^{\mu} z'^{\nu} = 0 .$$

So altogether for the vacuum functional $\Psi = e^{\int f_{\mu\nu} z'^{\mu} z'^{\nu}}$ the commutator up to the first order becomes

$$\frac{1}{4} \left[\int d\sigma \mathcal{H}, \int d\sigma \sigma^k \mathcal{H} \right] \Psi = \int d\sigma k \sigma^{k-1} \mathcal{P} \Psi . \quad (23)$$

No additional term results! This could be derived, more easily, by substituting $f_{\mu\nu} = a g_{\mu\nu}$, which is the most general form the first term could have, in the expansion of the logarithm of the vacuum functional, for an appropriate constant a . From (23) we deduce that in addition

to the vacuum state, all the excited states will not contribute to the commutator (20) at the first order as f is general. We can calculate the central charge term appearing when we use a second order test functional $F = \int f_{\mu\nu} \mathcal{D}z'^\mu \mathcal{D}z'^\nu$. The action of M_k on F is

$$\begin{aligned}
M^k \int f_{\mu\nu} \mathcal{D}z'^\mu \mathcal{D}z'^\nu = & \\
& \int d\sigma \sigma^k \left\{ \frac{2b_0^2}{\sqrt{s}} f_\nu^\nu + \frac{4b_0^1}{\sqrt{s}^3} D^\nu f_{\nu\mu} \mathcal{D}z'^\mu + \frac{1}{\sqrt{s}} (\bar{J}_2 f)_{\mu\nu} \mathcal{D}z'^\mu \mathcal{D}z'^\nu \right\} + \\
& \int d\sigma \sigma^{k-2} k(k-1) \left\{ \frac{b_0^0}{\sqrt{s}} D^\nu f_{\nu\rho_2} \mathcal{D}z'^{\rho_2} + \frac{2b_0^0}{\sqrt{s}} f_{\kappa\nu} z'^\rho z'^\gamma R_{\gamma\rho}^\nu{}^\kappa - \frac{b_0^1}{\sqrt{s}^3} f_\nu^\nu \right\} + \\
& \int d\sigma \sigma^{k-4} \frac{k(k-1)(k-2)(k-3)}{8} \frac{b_0^0}{\sqrt{s}} f_\nu^\nu + \int d\sigma k \sigma^{k-1} \frac{4b_0^0}{\sqrt{s}} R_{\gamma}{}^\rho f_{\kappa\rho} z'^\gamma \mathcal{D}z'^\kappa, \tag{24}
\end{aligned}$$

where

$$(\bar{J}_2 f)_{\mu\nu} = b_0^0 \bar{\Delta} f_{\mu\nu} + 2b_0^0 R_\mu^\rho f_{\nu\rho}. \tag{25}$$

Using the relation

$$M^k \int f_\rho \mathcal{D}z'^\rho = \dots + \int d\sigma \left\{ k \sigma^{k-1} \frac{2b_0^0}{\sqrt{s}} R_\gamma^\rho f_\rho z'^\gamma + \frac{k(k-1)}{2} \sigma^{k-2} \frac{b_0^0}{\sqrt{s}} D^\nu f_\nu \right\} \tag{26}$$

we can show that the commutator $(\Delta M^k - M^k \Delta)F$ is

$$\begin{aligned}
(\Delta M^k - M^k \Delta)F = & \\
& \int d\sigma \sigma^{k-2} k(k-1) \left\{ -\frac{2b_0^0 b_0^1}{s^2} [D^\nu D^\mu f_{\mu\nu} + R^{\nu\kappa} f_{\nu\kappa}] + \right. \\
& \frac{b_0^{02}}{s} [2R_{\gamma\rho} D^\nu f_{\nu\rho} z'^\gamma - \frac{1}{b_0^0} D^\nu (\bar{J}_2 f)_{\nu\rho} \mathcal{D}z'^\rho + \frac{1}{a^4} (f_{\gamma\rho} + (N-2) f_\nu^\nu g_{\gamma\rho}) z'^\gamma z'^\rho] \left. \right\} - \\
& \int d\sigma \sigma^{k-4} \frac{k(k-1)(k-2)(k-3)}{4} \frac{b_0^{02}}{s} R^{\nu\rho} f_{\nu\rho}. \tag{27}
\end{aligned}$$

This is non-zero even for $f_{\mu\nu} = g_{\mu\nu}$.

VI. MODIFICATION OF THE CENTRAL CHARGE TERM

Let us ignore regularization problems and define the generalized Virasoro operators as

$$\begin{aligned}
L[u] &= \frac{1}{4} \int u(\sigma) (P^\mu P_\mu + z'^\mu z'_\mu - z'^\mu P_\mu - P^\mu z'_\mu) d\sigma \\
\tilde{L}[u] &= \frac{1}{4} \int u(-\sigma) (P^\mu P_\mu + z'^\mu z'_\mu + z'^\mu P_\mu + P^\mu z'_\mu) d\sigma,
\end{aligned}$$

where the operator \tilde{L} is the same as L , but with σ replaced by $-\sigma$. With these definitions we can express H , L^k and P in terms of L and \tilde{L} as

$$L[1] + \tilde{L}[1] = \frac{1}{2} \int (P^\mu P_\mu + z'^\mu z'_\mu) = H$$

$$\begin{aligned}
L[1] - \tilde{L}[1] &= -\frac{1}{2} \int (z'^\mu P_\mu - \frac{1}{2} z'^\mu P_\mu) = -P \\
L[\sigma] + \tilde{L}[\sigma] &= - \int \sigma z'^\mu P_\mu = - \int \sigma \mathcal{P} \\
L[\sigma] - \tilde{L}[\sigma] &= \frac{1}{2} \int \sigma (P^\mu P_\mu + z'^\mu z'_\mu) = L^1
\end{aligned}$$

and so on. Using these identifications we can see how an extension to the algebra (23), will be placed in the modified Virasoro algebra. Let us extend the algebra with the additional term A , as

$$[L[1], L[\sigma^k]] = -L[1, \sigma^k] + A_k$$

where

$$[u, v] = uv' - u'v \Rightarrow [1, \sigma^k] = k\sigma^{k-1}$$

and

$$[\tilde{L}[1], \tilde{L}[\sigma^k]] = -\tilde{L}[1, \sigma^k] + \tilde{A}_k$$

We separate two cases: $k = 2l$ and $k = 2l + 1$. Then

$$\begin{aligned}
L[\sigma^{2l}] &= \frac{1}{4} \int \sigma^{2l} (P^2 + z'^2 - z'P - Pz') \\
\tilde{L}[\sigma^{2l}] &= \frac{1}{4} \int \sigma^{2l} (P^2 + z'^2 + z'P + Pz')
\end{aligned}$$

so that

$$L[\sigma^{2l}] + \tilde{L}[\sigma^{2l}] = \frac{1}{2} \int \sigma^{2l} (P^2 + z'^2) = \int \sigma^{2l} \mathcal{H}$$

and similarly

$$\begin{aligned}
L[\sigma^{2l+1}] &= \frac{1}{4} \int \sigma^{2l+1} (P^2 + z'^2 - z'P - Pz') \\
\tilde{L}[\sigma^{2l+1}] &= -\frac{1}{4} \int \sigma^{2l+1} (P^2 + z'^2 + z'P + Pz')
\end{aligned}$$

so that

$$L[\sigma^{2l+1}] - \tilde{L}[\sigma^{2l+1}] = \frac{1}{2} \int \sigma^{2l+1} (P^2 + z'^2) = \int \sigma^{2l+1} \mathcal{H}.$$

For the first case we have

$$\begin{aligned}
\left[\int \mathcal{H}, \int \sigma^{2l} \mathcal{H} \right] &= [L[1] + \tilde{L}[1], L[\sigma^{2l}] + \tilde{L}[\sigma^{2l}]] = \\
&= -L[1, \sigma^{2l}] + A_{2l} - \tilde{L}[1, \sigma^{2l}] + \tilde{A}_{2l} = \\
&= 2l \int \sigma^{2l-1} z'^\mu P_\mu + A_{2l} + \tilde{A}_{2l},
\end{aligned}$$

while for the second

$$\begin{aligned}
\left[\int \mathcal{H}, \int \sigma^{2l+1} \mathcal{H} \right] &= [L[1] + \tilde{L}[1], L[\sigma^{2l+1}] - \tilde{L}[\sigma^{2l+1}]] = \\
&= -L[1, \sigma^{2l+1}] + A_{2l+1} + \tilde{L}[1, \sigma^{2l+1}] - \tilde{A}_{2l+1} = \\
&= (2l+1) \int \sigma^{2l} z'^\mu P_\mu + A_{2l+1} - \tilde{A}_{2l+1}.
\end{aligned}$$

If we compare the outcome of the commutator $[f \mathcal{H}, f \sigma^k \mathcal{H}]$ acting on a general functional, then we can identify the quantities A and \tilde{A} . For the functional $\int f_{\mu\nu} z'^\mu z'^\nu$ it gives $A = \tilde{A} =$

0. Also, if we set $\tilde{A}_{2l} = A_{2l}$ and $\tilde{A}_{2l+1} = -A_{2l+1}$ we have for the functional $\int f_{\mu\nu} \mathcal{D}z'^\mu \mathcal{D}z'^\nu$ that

$$\begin{aligned} \tilde{A}_k = A_k = & \frac{1}{2} \int d\sigma \sigma^{k-2} k(k-1) \left\{ -\frac{2b_0^0 b_0^1}{s^2} [D^\nu D^\mu f_{\mu\nu} + R^{\nu\kappa} f_{\nu\kappa}] + \right. \\ & \frac{b_0^{02}}{s} \left[2R_{\gamma\rho} D^\nu f_{\nu\rho} z'^\gamma - \frac{1}{b_0^0} D^\nu (\bar{J}_2 f)_{\nu\rho} \mathcal{D}z'^\rho + \frac{1}{a^4} (f_{\gamma\rho} + (N-2)f_\nu^\nu g_{\gamma\rho}) z'^\gamma z'^\rho \right] \Big\} - \\ & \frac{1}{2} \int d\sigma \sigma^{k-4} \frac{k(k-1)(k-2)(k-3)}{4} \frac{b_0^{02}}{s} R^{\nu\rho} f_{\nu\rho} . \end{aligned} \quad (28)$$

Finally, we note that the central charge we arrived at in Section IV was a result of the non locality of the vacuum state. This is described by the function $H(\sigma, \sigma')$, which in contrast to the usual delta function (appearing when we use local functionals), brings in the non-local character of the vacuum and gives the specific form to the central charge term. However, as we have calculated, we get operator-like terms for the central charge of the algebra on S^N which is a fingerprint of the generated mass.

VII. A ‘WRONG’ BUT INSTRUCTIVE EXAMPLE

Let us consider the algebra of the usual Virasoro operators, but acting on massive states, rather than on massless states of a conformal theory. For this we will take the free, massive scalar field theory, while the Virasoro operators, $L[u]$, will be the ones defined for the string. We will choose $u(\sigma) = \sigma^k$ and for the state to be the vacuum wave functional

$$\Psi[z] = \exp - \int d\sigma z \sqrt{-\partial^2 + m^2} z . \quad (29)$$

Using the general result of equation (21) we are interested in the first term of (20) as the rest give the usual terms of the algebra apart from the anomaly ones. For an expansion of the logarithm of (29) valid for slowly varying fields, z , of the form

$$z \sqrt{-\partial^2 + m^2} z = z m \left(1 - \frac{\partial^2}{2m^2} - \frac{(\partial^2)^2}{8m^4} + \dots \right) z \quad (30)$$

we have

$$\begin{aligned} M^k \Psi[z] = & \iint d\sigma d\sigma' \left(\frac{\sigma + \sigma'}{2} \right)^k G_s(\sigma, \sigma') \frac{\delta}{\delta z(\sigma)} \frac{\delta}{\delta z(\sigma')} \Psi[z] = \\ & \int d\sigma m^2 \left[\left(b_0^0 \sigma^k - b_0^0 \frac{k(k-1)}{4m^2} \sigma^{k-2} - \frac{b_0^1}{s} \frac{\sigma^k}{m^2} - b_0^0 \frac{k(k-1)(k-2)(k-3)}{64m^4} \sigma^{k-4} \right. \right. \\ & \left. \left. - \frac{3}{2} \frac{b_0^1}{s} \frac{k(k-1)}{4m^4} \sigma^k - \frac{b_0^2}{s^2} \frac{\sigma^k}{4m^4} \right) z^2 + b_0^0 \frac{\sigma^k}{4m^4} z'^2 + \dots \right] \Psi[z] + (\text{const.}) \times \Psi[z] \end{aligned} \quad (31)$$

so that

$$\begin{aligned}
\Delta M^k \Psi[z] &= \iint d\sigma d\sigma' G_s(\sigma, \sigma') \frac{\delta}{\delta z(\sigma)} \frac{\delta}{\delta z(\sigma')} \times \\
&\iint d\sigma d\sigma' \left(\frac{\sigma + \sigma'}{2} \right)^k G_s(\sigma, \sigma') \frac{\delta}{\delta z(\sigma)} \frac{\delta}{\delta z(\sigma')} \Psi[z] = \\
&\int d\sigma m^2 \left[2 \frac{b_0^0}{\sqrt{s}} \left(b_0^0 \sigma^k - b_0^0 \frac{k(k-1)}{4m^2} \sigma^{k-2} - \frac{b_0^1}{s} \frac{\sigma^k}{m^2} - b_0^0 \frac{k(k-1)(k-2)(k-3)}{64m^4} \sigma^{k-4} \right. \right. \\
&\quad \left. \left. - \frac{3}{2} \frac{b_0^1}{s} \frac{k(k-1)}{4m^4} \sigma^{k-2} - \frac{b_0^2}{s^2} \frac{\sigma^k}{4m^4} \right) + 2 \frac{b_0^2 b_0^0}{\sqrt{s}} \frac{\sigma^k}{4m^4} + \dots \right] \Psi[z] \\
&+ (\text{terms with } z) \times \Psi[z]
\end{aligned} \tag{32}$$

as well as

$$\begin{aligned}
M^k \Delta \Psi[z] &= \iint d\sigma d\sigma' \left(\frac{\sigma + \sigma'}{2} \right)^k G_s(\sigma, \sigma') \frac{\delta}{\delta z(\sigma)} \frac{\delta}{\delta z(\sigma')} \times \\
&\iint d\sigma d\sigma' G_s(\sigma, \sigma') \frac{\delta}{\delta z(\sigma)} \frac{\delta}{\delta z(\sigma')} \Psi[z] = \\
&\int d\sigma m^2 \left[2 \frac{b_0^0}{\sqrt{s}} \left(b_0^0 - \frac{b_0^1}{m^2 s} - \frac{b_0^2}{4m^4 s^2} \right) \right. \\
&\quad \left. + \frac{b_0^0}{2m^4} \left(\frac{b_0^0}{\sqrt{s}} \frac{k(k-1)(k-2)(k-3)}{16} \sigma^{k-4} + \frac{3}{2} \frac{b_0^1}{\sqrt{s}^3} k(k-1) \sigma^{k-2} + \frac{b_0^2}{\sqrt{s}^5} \sigma^k \right) + \dots \right] \Psi[z] \\
&+ (\text{terms with } z) \times \Psi[z].
\end{aligned} \tag{33}$$

Finally, their difference is

$$\begin{aligned}
(\Delta M^k - M^k \Delta) \Psi[z] &= \\
&\int d\sigma m^2 \left(-\frac{b_0^{02}}{m^4 \sqrt{s}} \frac{k(k-1)(k-2)(k-3)}{16} \sigma^{k-4} - \frac{3}{2} \frac{b_0^0 b_0^1}{m^4 \sqrt{s}^3} k(k-1) \sigma^{k-2} \right. \\
&\quad \left. - \frac{b_0^{02}}{2m^2 \sqrt{s}} k(k-1) \sigma^{k-2} + \dots \right) \Psi[z] \\
&+ (\text{terms with } z) \times \Psi[z].
\end{aligned} \tag{34}$$

The integral $\int d\sigma \sigma^r$ diverges as $1/\sqrt{s}^{r+1}$, for r a positive integer, with a certain regularization. Now we see how the mass appears explicitly. This, compared with relation (27), gives us a taste of the way we expect the mass to appear in the central charge term of the $O(N)$ σ -model.

The power series in $1/\sqrt{s}$ in (28) or (34) seems to suggest divergences of increasing order as $s \rightarrow 0$. But the expansion of the vacuum functional and of the kernel, $G_s^{\mu\nu}$, is consistent for large s , which is equivalent to small momentum. By analyticity arguments the series can be related to small s with resummation procedures (see [9]) from where we

can extract the correct divergence behavior of the central charge term. The resumed series should give a result which is independent of the regularization procedure if we include enough terms in the power series. In other words, a different point splitting kernel satisfying the proper transformation properties and initial conditions, should give the same answer after resummation is used. Thus, in order to get an improved approximation we need more terms in the expansion of the kernel beyond the second order. This can be achieved, for example, by the use of computer programs.

Finally, we saw above that in the lowest order in our expansion scheme the central charge, A_k , is zero. To find its exact form in the next order we need to know what $f_{\mu\nu}$ is. Apart from deriving A_k by solving the Schrödinger equation for $f_{\mu\nu}$ (see [7]), it can also be calculated from a renormalization group study of correlators of stress-energy tensor products [3]. Thus, knowing the particular form of the central charge term as given in (28) we can extract the form of $f_{\mu\nu}$ without having to solve the Schrödinger equation, resulting to the construction of the vacuum wave functional. In addition we believe that having this form of the central charge in hand would make easier the exact evaluation of it with another method.

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